Path Group in gauge theory and gravity*

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Abstract

Applications of the Path Group (consisting of classes of continuous curves in Minkowski space-time) to gauge theory and gravity are reviewed. Covariant derivatives are interpreted as generators of an induced representation of Path Group. Non-Abelian generalization of Stokes theorem is naturally formulated and proved in terms of paths. Quantum analogue of Equivalence Principle is formulated in terms of Path Group and Feynman path integrals.

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Contents

1 Introduction

One of the most important concepts in the modern quantum field theory is gauge field. This concept was introduced [1] on the basis of gauge symmetry, i.e. invariance under localized (depending on space-time points) symmetry groups. Later it became clear that gauge theory may be naturally formulated on the basis of such mathematical formalism as connections in fiber bundles and their curvatures. Gravity was from the very beginning formulated as theory of curved (pseudo-)Riemannian spaces. Fiber bundles turned out to be also efficient mathematical formalism for gravitational fields.

Here we shall give a short survey of an alternative mathematical background for both gauge theory and gravity, namely Path Group ([2, 3, 4, 5], see [6, 7] for reviews). Path Group (PG) is a generalization of translation group differing from the latter in that it may be applied to particles in external gauge and gravitational fields.

Formally Path Group may be defined as a set of certain classes of continuous curves in Minkowski space (generalization on the case of paths in an arbitrary group space is possible). Concept of PG came up as development of the Suvegesh's groupoid of parallel transports [8, 9] (a groupoid differs

from a group in that not any pair of its elements may be multiplied). The goal was to find a universal group such that parallel transports in various space-times be its representations.

The concept of PG may be obtained also in the attempt to globalize infinitesimal translations of a tangent space to a curved space-time. PG arises then instead of the usual translation group since curvature makes translations in different directions not commutative (see [10, 11, 12, 13] for other types of non-commutative translations).

PG reduces geometry to algebra: various geometries (gauge and gravitational fields) are nothing else than representations of the universal PG. In case of gauge fields the representation is simpler in that Lorentz group may be factorized out. In case of gravitational fields both Lorentz and PG (united to give the generalized Poincaré group) essencially participate in the constructions.

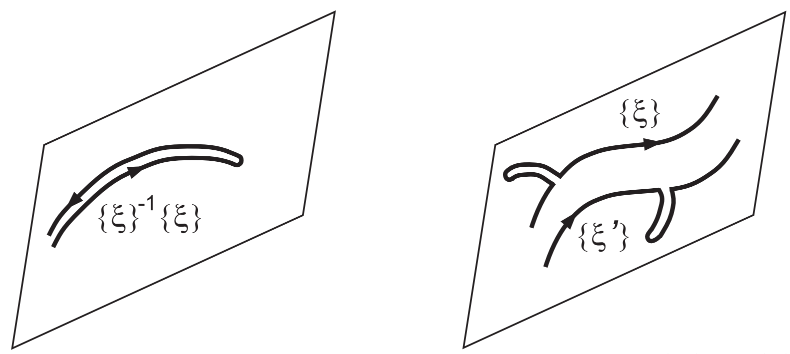
The natural character of PG is seen from the facts that it allows 1) to give a group-theoretical interpretation of covariant derivatives (both for gravitational or/and gauge fields) as generators of relevant representations of PG, 2) to formulate and prove a non-Abelian version of Stokes theorem and 3) to reduce the path integral in a curved space-time to the path integral in the flat space (a quantum version of Equivalence Principle)

2 Path Group

An element of Path Group (PG) is defined as a *class of curves* in Minkowski space constructed in such a way that the classes form a group. For curves in Minkowski space, $\{\xi\} = \{\xi(\tau) \in \mathcal{M}\}$, operation of multiplication $\{\xi'\}\{\xi\}$ may be naturally defined as passing of two curves one after another and inversion $\{\xi\}^{-1}$ as passing the same curve in the opposite direction. However, the set of all continuous curves is not, in respect to these operation, a group: 1) not all pairs of curves may be multiplied to give a continuous curve, 2) multiplication (when defined) is not assotiative and 3) product of a curve by its inverse does not yield a unit element (i.e. such one that multiplication by it does not change an arbitrary curve).

To correct the second defect, one may consider differently parametrized curves to be equivalent (and include them in the same class). To correct the third defect, one may consider equivalent those curves which differ by

inclusion of ‘appendices’ of the form $\{\xi\}^{-1}\{\xi\}$ (and go over to else wider classes of curves). At last, the first defect is overcome if we include in the same class those curves which differ by general shift: $\xi'(\tau) = \xi(\tau) + a$ (see Figure).



‘Null’ curve Equivalent curves $\{\xi'\} = \{\xi\}$

The resulting class of curves is called a *path* and denoted by p or $[\xi]$. All paths form *Path Group* P. A path may be presented by any curve from the corresponding class.

If the curves which differ by general shift are not considered equivalent, we have more narrow classes. The end points are the same for all curves in such a class. It may be therefore called a *pinned path* and denoted as $\hat{p} = p_x^{x'} = [\xi]_x^{x'}$ (or p_x) where x is the initial and x' final point of any curve presenting the given pinned path.

All pinned paths form a groupoid $\hat{P}$ since not any pair of pinned paths may be multiplied. Pinned paths are often convenient for constructing representations of the group P of free paths. It is important that the pair p, x of a free path and a point unambiguously determines the pinned path p_x starting in x and having the same shape as p.

Paths may be defined [6, 7] for an arbitrary group space G leading to the group of paths $P(G)$ in G (the role of general shift is played in this case by right shift in the group). We shall restrict ourselves by considering only the path group in Minkowski space, $P = P(\mathcal{M})$. This will prove to be sufficient for applications to gauge theory and gravity.

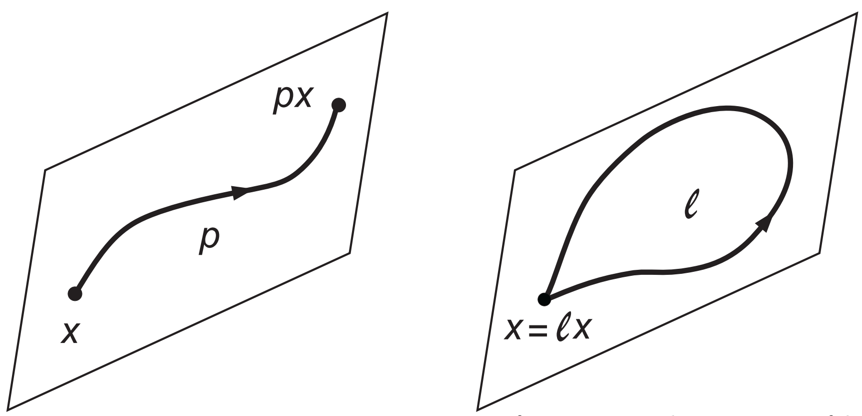
3 Gauge fields as representations of PG

Theory of free elementary particles is governed by the group of translations. Starting from Path Group instead of the translation group we obtain theory of particles in an external gauge field. Gauge fields thus arise independently of the idea of gauge symmetry.

The key point for constructing theory of particles is that Path Group P (generalized translations) acts on Minkowski space $\mathcal{M}$ transitively. An important consequence is that particles have to be described by the induced representations of Path Group.

3.1 Group and localization

Theory of elementary particles may be constructed starting from a certain group and implying the requirement of locality [14, 15, 6] (see [16] for an analogous construction). In our case Path Group P will play the role of the governing group and Minkowski space $\mathcal{M}$ the role of the localization space. Note that the action of the group P on the space $\mathcal{M}$ is naturally defined as shifting a point $x \in \mathcal{M}$ along the path $p \in P$ (see Figure). An arbitrary



Path acting on a point   Invariance of points under action of loops

point in $\mathcal{M}$ is invariant under the action of closed paths (loops). Therefore, the subgroup of loops $L \subset P$ is a stabilizer of an arbitrary point. This means that the space $\mathcal{M}$ may be presented as a quotient space $\mathcal{M} = P/L$.

If some point $\mathcal{O} \in \mathcal{M}$ is chosen as an origin in $\mathcal{M}$, then the point $x \in \mathcal{M}$ is identified with the coset $p'L \in P/L$ where $x = p'\mathcal{O}$. The space $\mathcal{M} = P/L$ will serve in our case as a localization space.

The general scheme of constructing theory of particles starting from the group and localization space is following [14, 15, 6].

Let theory of particles be governed by the group P (which may be symmetry or kinematic group) and characterized by localization in the space $\mathcal{M}$ (i.e. the subspace of states $\mathcal{H}_x$ is defined in which the particle is localized in an arbitrary point x of $\mathcal{M}$). Then the group P must act on the space $\mathcal{M}$ (so that $p:\ x \to px$ for $x \in \mathcal{M}$, $p \in P$), and this action must be in accord with the representation U of P acting in the space of the particle's states:

$$U(p)\mathcal{H}_x = \mathcal{H}_{px}.$$

A representation $U(P)$ possessing this property is called imprimitive.

Assume that the action of P on $\mathcal{M}$ is transitive (this may be done without loss of generality, because otherwise we may divide $\mathcal{M}$ in imprimitive subspaces). Then $\mathcal{M}$ is a homogeneous space and can be presented as a quotient space, $\mathcal{M} = P/L$, with an appropriate subgroup $L \subset P$.

The representation $U(P)$ acts then transitively on the set of subspaces $\{\mathcal{H}_x | x \in \mathcal{M} = P/L\}$. According to the imprimitivity theorem [17], such a representation (transitive but imprimitive) is equivalent to the representation induced from some representation $\alpha(L)$ of the subgroup L:

$$U(P) = \alpha(L) \uparrow P.$$

This gives a receipt for constructing theory of local particles. Knowing the group P and the space $\mathcal{M} = P/L$ of localization we can restore the representation $U(P) = \alpha(L) \uparrow P$ acting in the space of states $\mathcal{H}$ of the particle (and therefore can restore the space $\mathcal{H}$ itself). To do this, we have 1) to choose arbitrarily a representation $\alpha(L)$ of the subgroup L and 2) to induce it onto the whole group P.

Inducing a representation $\alpha(L)$ of a subgroup onto the whole group P may be achieved in the following way [17]. Vectors of the carrier space $\mathcal{H}$ of the induced representation $U(P) = \alpha(L) \uparrow P$ are presented by functions $\Psi(p)$ on P with values in the carrier space $\mathcal{L}_\alpha$ of $\alpha(L)$ with the additional *structure condition* imposed on the functions:

$$\Psi(pl) = \alpha(l^{-1})\Psi(p), \quad l \in L, p \in P$$

The induced representation acts on these functions by left shifts:

$$(U(p)\Psi)(p') = \Psi(p^{-1}p'), \quad p, p' \in P.$$

The representation $\alpha(L)$ is arbitrary in this construction. It describes *internal degrees of freedom* of the particle which it possesses even if its localization in $\mathcal{M}$ is fixed. States of the particle are described by the functions $\Psi(p)$ depending on elements of the group P. In our case these are paths, so we arrive to the non-local formalism of path-dependent wave functions. It is a group-theoretical version of Mandelstam's path-dependent fields [18, 19, 20].

However, despite of the non-local form, the theory must be essentially local because the requirement of locality was imposed from the very beginning. Therefore, *explicitly local form* of the representation must exist.

For constructing this form we need an extension $\alpha(P)$ of the representation $\alpha(L)$ onto the group P such that $\alpha(pl) = \alpha(p)\alpha(l)$ for arbitrary $p \in P$ and $l \in L$. Local wave functions $\psi(x)$ may be defined then as

$$\psi(x) = \alpha(p')\Psi(p')$$

where the point $x \in \mathcal{M}$ corresponds to the coset $p'L \in P/L$. Now the action of the induced representation is given by

$$(U(p)\psi)(x) = \alpha(p')[\alpha(p^{-1}p')]^{-1}\psi(p^{-1}x)$$

(for simplicity we denote this representation by the same letter although it is only equivalent but not identical to the preceding one).

3.2 Particles in gauge field

Applying the general scheme to the Path Group P, its subgroup of loops L and Minkowski space $\mathcal{M} = P/L$, we obtain that the representation $\alpha(L)$ describes a gauge field and the induced representation $U(P) = \alpha(L) \uparrow P$ presents a particle in this field. We shall illustrate this approach starting from the usual description of a gauge field by vector-potential.

Making use of a vector-potential $A_\mu(x)$, introduce a representation of the groupoid of pinned paths by ordered exponentials

$$\hat{\alpha}(\hat{p}) = \mathcal{P}\exp\left\{i\int_{\hat{p}} A_\mu(x)dx^\mu\right\}$$

(integration here is performed along any of the curves from the class $\hat{p}$). Fixing an (arbitrarily) point $\mathcal{O} \in \mathcal{M}$ as an origin of $\mathcal{M}$, we may associate a pinned paths $p_{\mathcal{O}}$ (starting in $\mathcal{O}$) with any free path $p \in P$. In these notations, the representation of loops may be expressed as $\alpha(l) = \hat{\alpha}(l_{\mathcal{O}})$ and the expansion of this representation onto the whole group P (with the properties specified above) as $\alpha(p) = \hat{\alpha}(p_{\mathcal{O}})$.

This determines a local form of the representation $U(P) = \alpha(L) \uparrow P$ which is given by the following elegant formula:

$$(U(p)\psi)(x) = \hat{\alpha}(p^x_{x'})\psi(x').$$

Another very convenient form of the same (or rather equivalent) representation may be expressed in terms of only free paths:

$$U(p) = \mathcal{P} \exp\left\{ -\int_p d\xi^\mu \nabla_\mu \right\} \quad \text{where} \quad (\nabla_\mu \psi)(x) = \left(\frac{\partial}{\partial x^\mu} - iA_\mu(x) \right).$$

The covariant derivatives are therefore generators of the representation $U(P)$ of Path Group (just as ordinary derivatives are generators of translations). By this the *group-theoretical interpretation of covariant derivatives* is given.

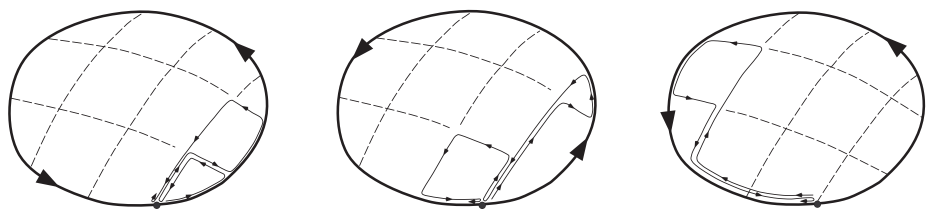
The representation of the group of loops, $\alpha(L)$, provides in this scheme a non-local description of a gauge field. This description is better than local descriptions by vector-potential and by field strength. Indeed, the first of these descriptions is redundant since various vector-potentials may correspond to the same physical situation, and the second is insufficient because non-local topological effects (such as Aharonov-Bohm effect) cannot be described by field strength. The 'intermediate' non-local description by $\alpha(L)$ is adequate.

The scheme presented here leads to gauge fields without usage of gauge transformations and the idea of gauge invariance. *Gauge transformations* arise in this scheme as presentation of *arbitrariness in the local description* of gauge fields by vector-potentials. Let us change the vector-potential A_μ to A'_μ in such a way that the representation of L does not change, $\alpha'(l) = \alpha(l)$, i.e. the non-local path-dependent description of the gauge field is the same. Then the new and old vector-potentials may be shown to be connected by a gauge transformation.

3.3 Non-Abelian Stokes theorem

One more evidence of natural character of the non-local path-dependent presentation of gauge fields is the non-Abelian version of Stokes theorem [5, 6].

Any loop may be presented as a *product of small 'lasso'* of the form $p^{-1}\,\delta l\,p$ with δl being a very small loop and p finite path (see Figure). The product



is then of the form $l = \mathcal{P}\prod_j p_j^{-1}\,\delta l_j\,p_j$. From the multiplicative properties of the representation $\alpha(L)$ we have

$$\alpha(l) = \mathcal{P}\prod_j [\alpha(p_j)]^{-1}\hat{\alpha}\left((\delta l_j)_{x_j}\right)\alpha(p_j) = \mathcal{P}\prod_j \exp\left(\frac{i}{2}\mathcal{F}_{\mu\nu}(p_j)\sigma_j^{\mu\nu}\right)$$

where $x = p\,\mathcal{O}$ is a point which the path p brings the origin $\mathcal{O}$ to, δl_x is a pinned loop having the shape δl and starting in x, and $\mathcal{F}_{\mu\nu}(p) = [\alpha(p)]^{-1}F_{\mu\nu}(p\mathcal{O})\alpha(p)$. Symbolically this may be presented in the form

$$\mathcal{P}\exp\left(i\int_{\partial\Sigma} A_\mu(x)\,dx^\mu\right) = \mathcal{P}\exp\left(\frac{i}{2}\int_\Sigma F_{\mu\nu}(x)\,dx^\mu\wedge dx^\nu\right)$$

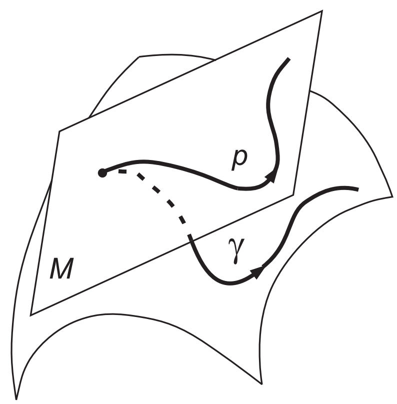
which is an ordered-exponential form of the non-Abelian Stokes theorem. Other forms of this theorem are in [21, 22].

4 Paths in gravity

Path Group may be used for description of a gravitational field (curved space-time) and particles in a gravitational or gauge + gravitational field [6, 7]. Minkowski space plays the role of a standard tangent space, connected with the curved space-time in a nonholonomic way.

4.1 Flat models of curves and fiber bundle of frames

There is a natural (but non-holonomic) mapping [2, 23] of the curves (paths) in the tangent space onto the curves in the curved space-time $\mathcal{X}$ (see Figure). The necessary relations may be presented in terms of the fiber bundle $\mathcal{B}$ (over



$\mathcal{X}$) consisting of local frames i.e. bases $b = \{b_\alpha | \alpha = 0, 1, 2, 3\}$ of tangent spaces in the points $x \in \mathcal{X}$. Coordinates in $\mathcal{B}$ are $\{x^\mu, b^\mu_\alpha\}$.

The key instrument for this end are (horizontal) *basis vector fields* in $\mathcal{B}$:

$$B_\alpha = b^\mu_\alpha \left(\frac{\partial}{\partial x^\mu} - \Gamma^\lambda_{\mu\nu}(x)\, b^\nu_\beta \frac{\partial}{\partial b^\lambda_\beta} \right)$$

The *representation of Path Group* by operators acting on functions in $\mathcal{B}$ as well as the *action of* P *on* $\mathcal{B}$ are readily defined:

$$U(p) = \mathcal{P} \exp \left(\int_p d\xi^\alpha \, B_\alpha \right), \quad (U(p)\psi)(b) = \psi(bp).$$

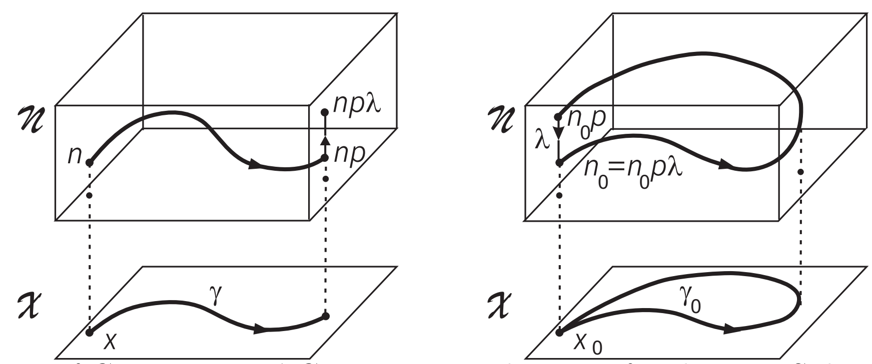
Thus defined mapping $p : \; b \to bp$ is a parallel transport along the curve in $\mathcal{X}$ which corresponds to the path p in $\mathcal{M}$ in the sense of the above mentioned mapping.

If the connection conserves the metric, these operatins may be restricted on the *subbundle* $\mathcal{N} \in \mathcal{B}$ *of orthonormal frames* yielding operators $U(p)$ acting on functions in $\mathcal{N}$ and the mapping $p: \ n \to np$ such that

$$(U(p)\psi)(n) = \psi(np).$$

Thus, the action of Path Group P by *parallel transports* is defined on $\mathcal{N}$. Lorentz group Λ acts on $\mathcal{N}$ as the *structure group* of the fiber bundle, i.e. $\lambda: \ n \to n\lambda$ where $(n\lambda)^{\mu}_{\alpha} = n^{\mu}_{\beta}\lambda^{\beta}_{\alpha}$. Therefore the action of their semidirect product (generalized Poincaré group $Q = \Lambda \otimes P$) is defined.

More precisely, the elements of Q are of the form $q = p\lambda$ (where $p \in P$ and $\lambda \in \Lambda$), multiplication of them is determined by the relation $\lambda[\xi]\lambda^{-1} = [\lambda\xi]$, while the action of Q on fiber bundle $\mathcal{N}$ is defined as $nq = n\,p\lambda = (np)\lambda$ ($p: \ n \to np$ being a parallel transport and $\lambda: \ n \to n\lambda$ the action of the structure group). The resulting action of $p\lambda$ (Figure, left diagram) is in accord with the multiplication law in the group Q.



Action of Gen. Poincaré Group   Element of Holonomy Subgroup

4.2 Holonomy Subgroup

Holonomy Subgroup H of the generalized Poincaré group Q is defined in respect to the (arbitrarily chosen and fixed) local frame $n_0 \in \mathcal{N}$ as the subset leaving n_0 invariant (a stationary subgroup of n_0) so that $h = p\lambda \in H$ if $n_0\,p\lambda = n_0$ (see Figure, right diagram).

Holonomy Subgroup H in the generalized Poincaré group Q plays the role (for gravity) analogous to the subgroup of loops L in the Path Group P (for

gauge theory). This is why a representation $\alpha(H)$ represents a *gauge + gravitational field* while the induced representation $U(Q) = \alpha(H) \uparrow Q$ describes particles in this combined field. If the representation α is trivial, $\alpha(h) \equiv 1$, the induced representation describes particles in a pure gravitational field (corresponding to the Holonomy Subgroup H).

Holonomy Subgroup $H \subset Q$ determines geometry. The geometry (including topology) may be restored if the subgroup H is given. This reconstruction procedure may be applied [24] to explore geometry of hyperbolic, or Lorentzian, cones, starting from $H = \{l\nu^k | l \in L, k \in Z\}$ where ν is a fixed element of the Lorentz group.

4.3 Quantum Equivalence Principle

Various quantum analogues of the Einstein's Equivalence Principle may be defined. Validity of the quantum equivalence principle (QEP) depends of course on its definition. It is advantageous however to define QEP in such a way that it be valid. Path Group (together with the natural non-holonomic mapping of the curves in Minkowsky space onto the curves in the curved space-time) makes this possible [2, 3, 23] (see also [25, 26]).

A very brief formulation of QEP may be given as follows: evolution of a quantum particle in a curved space-time $\mathcal{X}$ must be described by the *same Feynman path integral* as in the flat space-time, if the paths in the standard tangent space $\mathcal{M}$ are used in the integral instead of the corresponding curves in the curved space-time $\mathcal{X}$.

Technically it is convenient to express QEP in terms of the representation $U(P)$ introduced above, putting $U(P)$ in the integrand of (flat) Feynman path integral instead of the ordinary translation. This may be done not only for purely gravitational but also for gravitational + gauge field (see [23] for details).

5 Conclusion

In the present paper we considered Path Group P (which generalizes translation group) and demonstrated the following issues:

1) Gauge fields are described by representations $\alpha(L)$ of the subgroup $L \subset P$ of loops. Particles in such a field are presented by the induced

representation $U(Q) = \alpha(L) \uparrow Q$. Non-Abelian Stokes theorem is naturally formulated and proved in terms of the representation $\alpha(L)$.

2) For application to gravity the generalized Poincaré group Q is necessary which is a semidirect product of Path Group by Lorentz group.

3) Geometry of a curved space-time (including non-trivial topology) is presented by the Holonomy Subgroup $H \in Q$.

4) Gauge + gravitational field is presented by the Holonomy Subgroup H together with its representation $\alpha(H)$ while particles in this field are described by the representation $\alpha(H) \uparrow Q$.

5) Quantum Equivalence Principle is naturally formulated in terms of Feynman path integral and natural non-holonomic mapping of curves in Minkowski space onto the curves in the curved space-time.

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